Problem A.12

Show that the rows and columns of a unitary matrix constitute orthonormal sets.

Solution

A unitary matrix is a matrix whose inverse is equal to its hermitian conjugate.

$$\mathsf{U}^{-1} = \mathsf{U}^\dagger$$

This means that the identity matrix is obtained whether one premultiplies U by $\mathsf{U}^\dagger$ or postmultiplies U by $\mathsf{U}^\dagger$.

$$\mathsf{U}^\dagger\mathsf{U} = \mathsf{I} \qquad\qquad \mathsf{U}\mathsf{U}^\dagger = \mathsf{I}$$

Consider element ij on both sides.

$$\left(\mathsf{U}^\dagger\mathsf{U}\right)_{ij} = \mathsf{I}_{ij} \qquad\qquad \left(\mathsf{U}\mathsf{U}^\dagger\right)_{ij} = \mathsf{I}_{ij}$$

$$\sum_{k=1}^n U^\dagger_{ik}U_{kj} = \delta_{ij} \qquad\qquad \sum_{k=1}^n U_{ik}U^\dagger_{kj} = \delta_{ij}$$

$$\sum_{k=1}^n \widetilde{U}^*_{ik}U_{kj} = \delta_{ij} \qquad\qquad \sum_{k=1}^n U_{ik}\widetilde{U}^*_{kj} = \delta_{ij}$$

$$\sum_{k=1}^n U^*_{ki}U_{kj} = \delta_{ij} \qquad\qquad \sum_{k=1}^n U_{ik}U^*_{jk} = \delta_{ij}$$

$$\sum_{k=1}^n U^*_{ki}U_{kj} = \delta_{ij} \qquad\qquad \sum_{k=1}^n U^*_{jk}U_{ik} = \delta_{ij}$$

$$U^*_{1i}U_{1j} + U^*_{2i}U_{2j} + \cdots + U^*_{ni}U_{nj} = \delta_{ij} \qquad\qquad U^*_{j1}U_{i1} + U^*_{j2}U_{i2} + \cdots + U^*_{jn}U_{in} = \delta_{ij}$$

This finite sum on the left is the inner product of the $n \times 1$ column vectors consisting of the elements in column i and column j of U. The finite sum on the right is the inner product of the $1 \times n$ row vectors consisting of the elements in row j and row i of U.

$$\langle C_i \,|\, C_j\rangle = \delta_{ij} \qquad\qquad \langle R_j \,|\, R_i\rangle = \delta_{ij}$$

$$\langle R_j \,|\, R_i\rangle^* = \delta^*_{ij}$$

$$\langle R_i \,|\, R_j\rangle = \delta_{ij}$$

Therefore, the rows and columns of a unitary matrix constitute orthonormal sets.